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# INCREASES IN RISK AND IN RISK AVERSION

BY

PETER A. DIAMOND and JOSEPH E. STIGLITZ

Number 97 -- January 1973

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# INCREASES IN RISK AND IN RISK AVERSION

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# Increases in Risk and in Risk Aversion

Peter A. Diamond and Joseph E. Stiglitz*

Analysis of individual behavior under uncertainty naturally focuses on the meaning and economic consequences of two statements:

1. one situation is riskier than another;
2. one individual is more risk averse than another.[1]

M. Rothschild and J. Stiglitz have considered one analysis of the first statement in terms of a change in the distribution of a random variable which keeps its mean constant and represents the movement of probability density from the center to the tails of the distribution. After restating their analysis in section 1, we consider an alternative definition of keeping constant the expectation of utility rather than the mean of the random variable.and analyze the effect of such an increase in risk. In section 3 we examine the concept of increased risk aversion which seems paired with the concept of increased risk and obtain sufficient conditions for the effect of increased risk aversion on choice to be of determinate sign. This approach follows that of K. Arrow and J. Pratt altered to fit our setting.

The second part of the paper applies these general results to specific problems, obtaining some well known results and some which are, to our knowledge, new. The purpose of the presentation of the well known results is to relate them to the general approach. For example, in section 5 we show that both decreasing absolute risk aversion and decreasing relative risk aversion can be obtained from the concavity definition of risk aversion, depending on whether the level of security holdings or the fraction of wealth held in securities is viewed as the control variable. The behavior of each control variable is then described by the matching index of risk aversion.

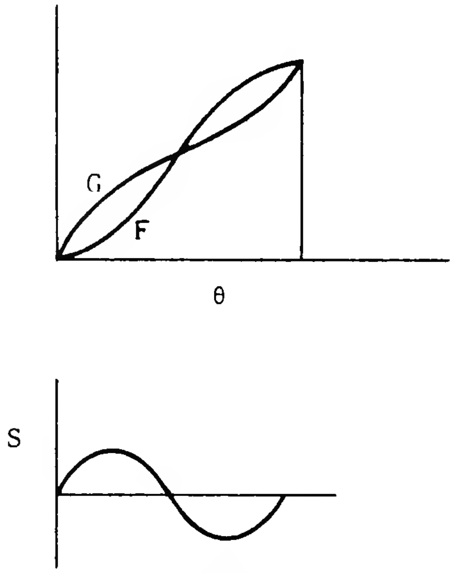
## 1. Mean Preserving Increase in Risk

### 1.1. Definition

Consider the distribution functions, F and G, of two random variables defined on the unit interval, and the difference between them

$$S(\theta) = G(\theta) - F(\theta) \tag{1}$$

If G is derived from F by taking weight from the center of the probability distribution and shifting it to the tails, while keeping the mean of the distribution constant, it is natural to say that G represents a riskier situation than F, and that the difference between these two variables is a mean preserving increase in risk. Illustrated in figure 1 is a simple example of such an increase where the distributions F and G cross only once (so it is unambigously clear that G has more weight in both tails). When this situation holds we shall say that S has the single crossing property and that S represents a simple mean preserving spread.



simple mean preserving spread

Figure 1

Analytically, we can characterize such a spread by the two conditions

$$\int_0^1 S(\theta)\, d\theta = 0 \tag{2}$$

There exists a $\theta$ such that

$$S(\theta) \leq (\geq)\ 0 \text{ when } \theta \geq (\leq)\ \hat{\theta} \tag{3}$$

The first condition assures us that the two distributions have the same mean;[2] the second, that there is a single crossing. An immediate implication of (2) and (3) is that the integral of S is nonnegative

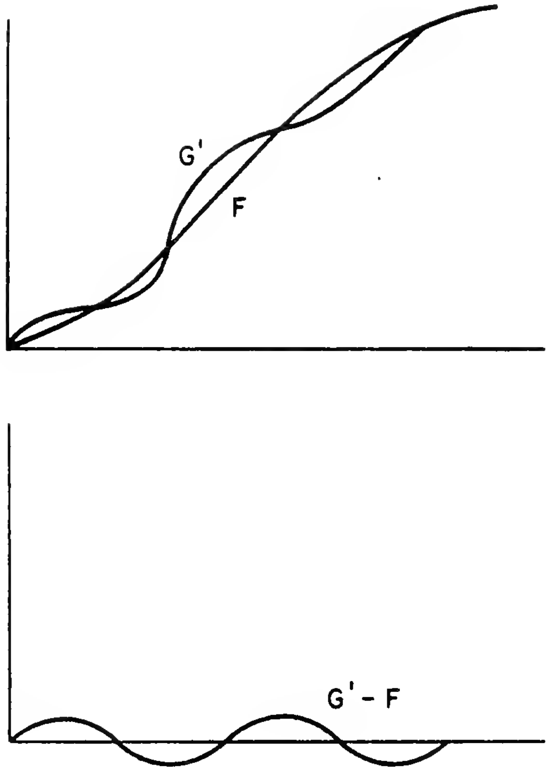
$$T(y) \equiv \int_0^y S(\theta)\, d\theta \geq 0 \qquad 0 \leq y \leq 1 \tag{4}$$

If we consider another distribution G' generated from G by a simple mean preserving spread, G' - F does not, in general, have the single crossing property, as can be seen in Figure 2. Since G' is riskier than G, and G is riskier than F, we would like to say that G' is riskier than F. Accordingly, (2) and (3) do not provide an adequate basis for a definition of "riskier". However, G' - F does satisfy conditions (2) and (4), as does the difference after any sequence of such steps. Rothschild and Stiglitz have shown, moreover, that if S(θ) satisfies conditions (2) and (4), S can be generated as a limit of a sequence of simple mean preserving spreads. Thus (2) and (4) provide a natural definition of increased risk.

Rothschild and Stiglitz have also shown that this definition of increased risk is equivalent to two other definitions: that all risk averters dislike increased risk, i.e.

$$\int_0^1 U(\theta)\, dF(\theta) \geq \int_0^1 U(\theta)\, dG(\theta) \quad \text{if} \quad U'' < 0$$

and that an increase in risk is the addition of noise to a random variable, i.e.



Mean preserving increase in risk

Figure 2

If X has distribution G, Y distribution F, there exists a random variable Z such

that $X = Y + Z$

with $E\{Z|Y\} = 0$

In the subsequent discussion, we shall consider a family of distributions;[3] $F(\theta,r)$, where increases in the shift parameter r represents increases in risk if $F_r(\theta,r)$ has the properties of $S(\theta)$ described above in (2) and (4).

## 1.2. Consequences

To consider the consequences of increased risk Rothschild and Stiglitz considered an expected utility maximizing individual whose utility depends on a random variable, $\theta$, and a control variable, $\alpha$

$$U = U(\theta,\alpha)$$

with the assumption $U_{\alpha\alpha} < 0$. Then, they related the optimal level of $\alpha$ to the level of the shift parameter r. In a form which will be useful for later analysis we can state their results as

Theorem 1: Let $\alpha^*(r)$ be the level of the control variable which maximizes $\int_0^1 U(\theta,\alpha)dF(\theta,r)$. If increases in r represent mean preserving increases in risk, (i.e., satisfy (2) and (4)) then $\alpha^*$ increases (decreases) with r if $U_\alpha$ is a convex (concave) function of $\theta$, i.e., if $U_{\alpha\theta\theta} > (<) 0$.

Proof: $\alpha^*(r)$ is defined implicitly by the first order conditon for expected utility maximization

$$\int_0^1 U_\alpha(\theta,\alpha)F_\theta(\theta,r)d\theta = 0 \tag{5}$$

Implicit differentiation of (5) gives us

$$d\alpha^*/dr = -[\int U_\alpha F_{\theta r}]/[\int U_{\alpha\alpha}F_\theta]$$

Since the denominator is negative $d\alpha^*/dr$ has the same sign as the numerator. Applying integration by parts twice (and **noting that** $F_r(0,r) = F_r(1,r) = T(0,r) = T(1,r) = 0$ ) we have

$$\int_0^1 U_\alpha F_{\theta r}d\theta = -\int_0^1 U_{\alpha\theta}F_r d\theta = \int_0^1 U_{\alpha\theta\theta}T(\theta,r)d\theta$$

where $T(\theta,r) = \int_0^\theta F_r(\theta,r)d\theta$. By (4) T is nonnegative so that $d\alpha^*/dr$ has the same sign as $U_{\alpha\theta\theta}(\alpha,\theta)$, assuming that $U_{\alpha\theta\theta}(\alpha,\theta)$ is uniformly signed for all $\theta$. ||

This theorem represents a complete characterization in the sense that changes in distributions not satisfying the definition of increasing risk can lead to decreases (increases) in $\alpha^*$ despite the convexity (concavity) of $U_\alpha$ and in the absence of convexity (concavity) of $U_\alpha$ increases in risk can lead to decreases (increases) in $\alpha^*$.

The approach of the next section will be to explore a similar analysis where increases in risk keep the mean of utility constant rather than the mean of the random variable. This is an advantage since some economic variables can naturally be described in several ways. For example, we could describe the consumption possibilities arising from a short term investment in a consol in terms of the interest rate or in terms of the future price of the consol. A change in riskiness of the investment which kept expected price constant will not keep the expected interest rate constant. Alternatively in an international trade setting with one export good, one import good, and uncertain terms of trade; increases in the riskiness of trade keeping the expected import price constant (with export price as numeraire) do not keep the expected export price constant (with import price as numeraire). More generally if we have a new random variable $\hat{\theta}$, monotonically related to the original random variable, $\hat{\theta} = \psi(\theta)$, then a change in the distribution of $\theta$ keeping its mean constant will generally change the mean of $\hat{\theta}$. In addition marginal utility of the control variable as a function of $\hat{\theta}$

$$\hat{U}_\alpha(\hat{\theta},\alpha) = U_\alpha(\psi^{-1}(\hat{\theta}),\alpha) \tag{6}$$

may not have the appropriate curvature to apply theorem 1 to changes in the distribution of $\hat{\theta}$, even if it is well behaved relative to $\theta$. By considering utility as the random variable, we obtain results which do not depend on the formulation of the problem in terms of $\theta$ rather than $\hat{\theta}$.

## 2. Mean Utility Preserving Increase in Risk

### 2.1. Definition

Let us denote by $\tilde{F}(u,\alpha,r)$ the distribution of $U(\theta,\alpha)$ induced by the distribution $F(\theta,r)$ when $\alpha$ is chosen; and by $\alpha^*(r)$, the optimal level of the control variable. (We normalize u so that it varies over the unit interval as $\theta$ does.) Expected utility can now be written as

$$\int_0^1 u \, d\tilde{F} = 1 - \int_0^1 \tilde{F} \, du \tag{7}$$

Then, we will say that increases in r correspond to mean utility preserving increases in risk if[4]

$$\tilde{T}(y,r) = \int_0^y \tilde{F}_r(u,\alpha^*(r),r)du \geq 0 \text{ for all } y \tag{8}$$

and

$$\tilde{T}(1,r) = \int_0^1 \tilde{F}_r(u,\alpha^*(r),r)du = 0 \tag{9}$$

Let us assume that U is monotonically increasing in $\theta$.[5] Then it is easy to relate the two definitions of increased riskiness. For any levels of u and $\alpha$ there is a unique level of $\theta$, which we denote by $U^{-1}(u,\alpha)$, which is defined by $u = U(\theta,\alpha)$. The distributions of u and of $\theta$ are now related by

$$\tilde{F}(U(\theta,\alpha),\alpha,r) = F(\theta,r) \tag{10}$$

or equivalently

$$\tilde{F}(u,\alpha,r) = F(U^{-1}(u,\alpha),r) \tag{11}$$

**By a change of variable**
we can now restate the conditions making up the definition of mean utility preserving increase in risk as

$$\tilde{T}(y,r) = \int_0^y U_\theta F_r(\theta,r)d\theta \geq 0 \text{ for all } y \tag{12}$$

$$\tilde{T}(1,r) = \int_0^1 U_\theta F_r(\theta,r)d\theta = 0 \tag{13}$$

(See figure 3 for an example of the relationship between F and $\tilde{F}$)

As with the mean preserving increase, condition (12) reflects a risk increase which is equivalent to the limit of a sequence of steps taking weight from the center of the probability distribution and shifting it to the tails. Now, however, expected utility, rather than the mean of the random variable is held constant. Thus it is natural to think of the mean utility preserving increase as a "compensated" adjustment of a mean preserving increase in risk.[6]

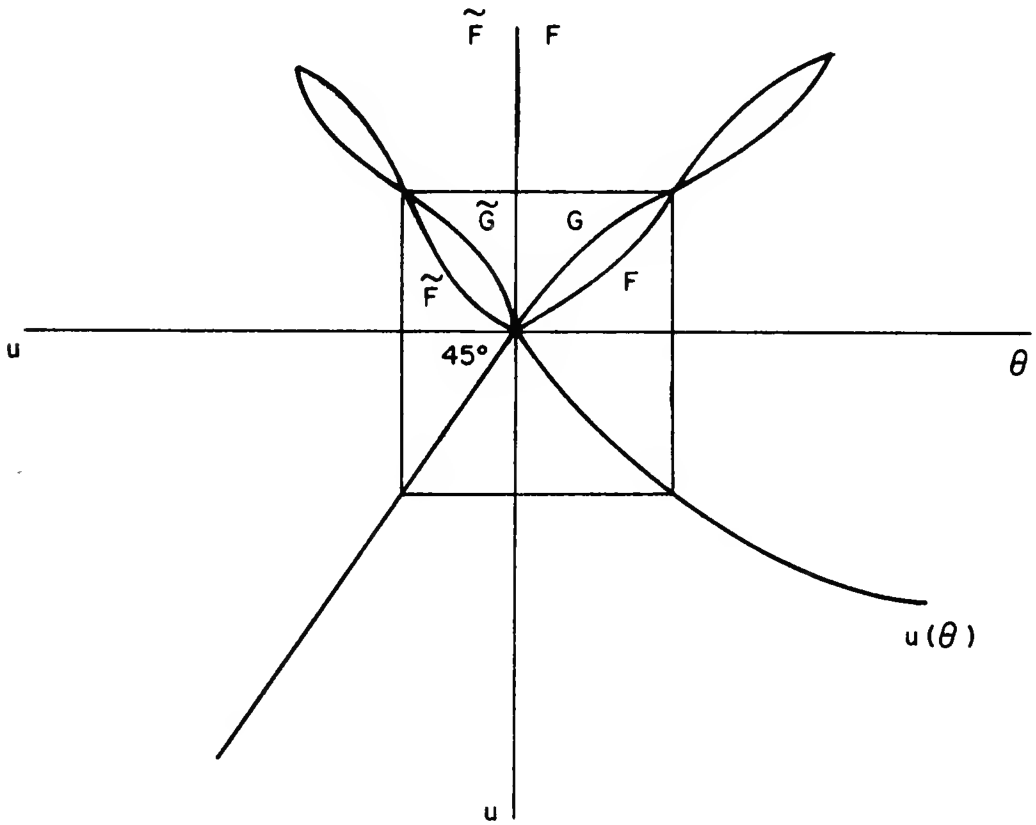


Figure 3

## 2.2. Consequence

We can now turn to the analogue to theorem 1. for this type of change in risk. We expect to find that the critical condition is the concavity of $U_\alpha$ as a function of u rather than $\theta$. Differentiating $U_\alpha(U^{-1}(u,\alpha),\alpha)$ with respect to u we have

$$\frac{\partial U_\alpha}{\partial u} = \frac{U_{\alpha\theta}}{U_\theta}, \quad \frac{\partial^2 U_\alpha}{\partial u^2} = \frac{U_{\alpha\theta\theta}}{U_\theta^2} - \frac{U_{\alpha\theta}U_{\theta\theta}}{U_\theta^3} \tag{14}$$

This last derivative can also be written as

$$U_\theta^{-1}\, \partial^2 \log U_\theta/\partial\theta\partial\alpha$$

With $U_\theta > 0$, we have $U_\alpha$ convex (concave) in u as

$$U_\theta U_{\alpha\theta\theta} - U_{\alpha\theta}U_{\theta\theta} > (<) 0 \tag{15}$$

Theorem 2:[7]

Let $\alpha^*(r)$ be the level of the control variable which maximizes $\int_0^1 U(\theta,\alpha)dF(\theta,r)$. If increases in r represent mean utility preserving increases in risk (i.e., satisfy (12) and (13)) then $\alpha^*$ increases (decreases) with r if $U_\alpha$ is a convex (concave) function of u, or (with $U_\theta > 0$)

$$(U_\theta U_{\alpha\theta\theta} - U_{\alpha\theta}U_{\theta\theta}) > (<) 0.$$

Proof:[8] As in Theorem 1, the sign of $d\alpha/dr$ is the same as the sign of

$$\int U_\alpha dF_r ( = \int U_\alpha F_{\theta r} d\theta.)$$

Applying integration by parts twice and noticing that

$F_r(0) = F_r(1) = \tilde{T}(0) = \tilde{T}(1) = 0$ we have

$$\int U_{\alpha} F_{\theta r} = -\int U_{\alpha\theta} F_r = -\int \frac{U_{\alpha\theta}}{U_\theta} U_\theta F_r = \int \frac{U_\theta U_{\alpha\theta\theta} - U_{\alpha\theta} U_{\theta\theta}}{U_\theta^2} \tilde{T} \qquad ||$$

This theorem represents a complete characterization in this case in the same sense that theorem 1 did for mean preserving increases.

From the two theorems, we see that a uniform sign of $U_{\alpha\theta\theta}$ will sign the response of $\alpha$ to a mean preserving change in risk while a uniform sign of $U_\theta U_{\alpha\theta\theta} - U_{\alpha\theta} U_{\theta\theta}$ will sign the response of $\alpha$ to a mean utility preserving change in risk. In the next section we will develop a notion of increases in risk aversion, which will allow us to interpret theorem 2 as stating that the *optimal response to a mean utility preserving increase in risk is to adjust the control variable so as to make U show less risk aversion.*

## 2.3. Choice of a distribution

The basis of theorem 2 is a set of sufficient conditions for signing the second derivative of expected utility with respect to the control and shift parameters. Since the order of differentiation doesn't matter, reversal of the role of shift and control variables still results in a sign - determined effect of shift variable on control variable.[9] Thus let us consider a situation where individuals select the distribution function of income (as when they select a career) and where the choice problem is parametrized by a variable which may reflect differences across people (such as risk aversion) or a level of some exogenous variable (such as the income tax rate). If distributions can be classed by riskiness (in the sense of the integral condition (12) or the single crossing property) and the shift parameter enters the utility function suitably we can sign the effect of the shift parameter (risk aversion or income tax rate, say) on the riskiness of selected careers.

More formally we consider an individual with utility function $U(\theta,\alpha)$ selecting among a family of distributions, $F(\theta,r)$ to maximize expected utility.

$$\max_{r} \int_0^1 U(\theta,\alpha)\, dF(\theta,r)$$

The first order condition for this maximization is

$$\int_0^1 U(\theta,\alpha)\ dF_r(\theta,r^*) = -\int_0^1 U_\theta F_r = -\int_0^1 \tilde{F}_r = 0 \qquad (16)$$

To apply the analysis of theorem 2, we must be able to show that

$\tilde{F}_r(\theta,r^*)$ satisfies (12) and (13). (13) is equivalent to the first order condition (16). (12) may be verified in the context of any particular problem, although it is more likely that one can readily verify the stronger single crossing property. Thus we state this result as:

Corollary: Let $r^*(\alpha)$ be the level of the control variable which maximizes $\int U(\theta,\alpha)\ d\tilde{F}(\theta,r)$. If there exists a $\hat{\theta}$ such that

$F_r(\theta,r^*)\ (\theta-\hat{\theta}) \leq 0$ for all $\theta$

then

$r^*$ increases (decreases) with $\alpha$ if $\dfrac{\partial^2 \log U_\theta}{\partial\theta\partial\alpha}$ is everywhere positive (negative)

In section 4 we consider two applications of this result.

## 3. Greater Aversion to Risk

### 3.1. Definition

Consider a mean utility preserving increase in risk for an individual. (We continue to assume that U increases in $\theta$.) If a second individual finds his expected utility decreasing from this change for any mean utility preserving increase in risk for the first individual, it is natural to say that the second individual is more risk averse than the first. It is also natural to say that a more risk averse individual will pay more for perfect insurance against any risk. Fortunately, as with riskiness, the different natural definitions of increased risk aversion are equivalent and lend themselves to an analysis of differences in behavior as a result of differences in risk aversion (either across individuals or as a result of a parameter change for a given individual). For some of our purposes it is convenient to work with a differentiable family of utility functions, $U(\theta,\rho)$, where $\rho$ represents an ordinal index of risk aversion. Given this notation we shall start by considering four equivalent definitions of increased risk aversion. Numbers two through four are, in our notation and setting, three of the five definitions which J. Pratt showed to be equivalent. The inference of number one from the others is due to H. Leland.

Theorem 3: The following definitions of the family of utility functions[10] $U(\theta,\rho)$ showing increasing risk aversion with the index $\rho$ are equivalent.[11]

(i) Mean utility preserving increases in risk are disliked by the more risk averse, i.e., for any change in a distribution of $\theta$ satisfying

$$\tilde{T}(y,r) = \int_0^y U_\theta(\theta,\rho)\, F_r(\theta,r)d\theta \geq 0 \text{ for all } y$$

and

$$\tilde{T}(1,r) = \int_0^1 U_\theta(\theta,\rho)\, F_r(\theta,r)d\theta = 0$$

we have

$$\int_0^1 U_\rho(\theta,\rho)F_{\theta r}(\theta,r)d\theta < 0 \; ; \tag{17}$$

(ii) For any risk, the risk premium for perfect insurance increases with risk aversion, i.e. for $p(\rho)$ defined by

$$U(\int \theta dF - p(\rho),\rho) = \int U(\theta,\rho)\, dF, p'(\rho) > 0 \quad ; \tag{18}$$

(iii) The measure of risk aversion increases with $\rho$, i.e.,

$$-\partial^2 \log U_\theta/\partial\theta\partial\rho > 0 \tag{19}$$

(iv) For each pair $(\rho_1,\rho_2)$ with $\rho_1 > \rho_2$, there exists a monotone concave function $\phi, \phi' > 0$, $\phi'' < 0$

such that

$$U(\theta,\rho_1) = \phi(U(\theta,\rho_2)). \tag{20}$$

Proof:

(i)<=> (iii) Applying integration by parts to (17) (and recalling that $F_r(0) = F_r(1) = \tilde{T}(0,r) = \tilde{T}(1,r) = 0$ ) we have

$$\int_0^1 U_\rho F_{\theta r} = -\int_0^1 U_{\theta\rho} F_r = -\int_0^1 \frac{U_{\theta\rho}}{U_\theta} U_\theta F_r = \int_0^1 \frac{\partial^2 \log U_\theta}{\partial\theta\partial\rho} \tilde{T}(\theta,r) \tag{21}$$

The negativity of the second derivative of $\log U_\theta$ and positivity of $\tilde{T}$ imply (17). Conversely, since any nonnegative $\tilde{T}$ is admissible positive values for the second derivative would permit a positive value for (17).

(iii) <=> (iv) Differentiating (20) with respect to $\theta$ we have

$$U_\theta(\theta,\rho_1) = \phi' U_\theta(\theta,\rho_2)$$

$$U_{\theta\theta}(\theta,\rho_1) = \phi'' U_\theta^2(\theta,\rho_2) + \phi' U_{\theta\theta}(\theta,\rho_2)$$

or solving for $\phi''$:

$$\phi'' = \left[ \frac{U_{\theta\theta}(\theta,\rho_1)}{U_\theta(\theta,\rho_1)} - \frac{U_{\theta\theta}(\theta,\rho_2)}{U_\theta(\theta,\rho_2)} \right] \frac{U_\theta(\theta,\rho_1)}{U_\theta^2(\theta,\rho_2)} \tag{22}$$

$$= \left[ \int_{\rho_2}^{\rho_1} (\partial^2 \log U_\theta/\partial\theta\partial\rho) d\rho \right] \frac{U_\theta(\theta,\rho_1)}{U_\theta^2(\theta,\rho_2)}$$

With $\phi''$ negative, for all $\rho_1 > \rho_2$, we obtain (iii) by taking the limit as $\rho_1$ goes to $\rho_2$. Conversely, the negativity of $\phi''$ is obtained by integration.

(ii) <= > (iii) Calculating $p'(\rho)$ by implicit differentiation of (18)

$$p'(\rho) = (U_\rho - \int U_\rho dF)/U_\theta \tag{23}$$

Considering the two terms in parantheses, we can express them as

$$U_\rho(\int\theta dF - p(\rho),\rho) = U_\rho(1,\rho) - \int_{\int\theta dF-p}^{1} U_{\theta\rho}d\theta \tag{24}$$

$$\int_0^1 U_\rho dF = U_\rho(1,\rho) - \int_0^1 U_{\theta\rho}Fd\theta \tag{25}$$

(expressing a difference as the integral of a derivative and using integration by parts). **Let $G(\theta)$ be 1 for $\theta$ between $\int\theta dF - p$ and 1 and zero otherwise. Then**

$$U_\rho - \int U_\rho dF = \int_0^1 U_{\theta\rho}(F - G)d\theta = \int_0^1 \frac{U_{\theta\rho}}{U_\theta} U_\theta(F-G)d\theta = -\int_0^1 \frac{\partial^2 \log U_\theta}{\partial\theta\partial\rho}\tilde{T}(\theta)d\theta \tag{26}$$

**since** $\int_0^1 U_\theta(F-G)d\theta$ is zero by the definition of the risk premium, (18).

The equivalence follows from (26) since $\tilde{T}$ is nonnegative and any distribution $F$ is admissible.||

It is interesting to note that (iii) permits us to conclude that with suitable normalization to keep means constant the more risk averse individual has a riskier distribution of log $U_\theta$. Specifically, behavior is unaltered by a multiplicative adjustment of utility, so we can use this degree of freedom **to equate** $\int\frac{\partial\log U_\theta(\theta,\rho)}{\partial\rho}dF$ with zero. The monotonicity of $\partial\log U_\theta/\partial\rho$ in $\theta$ then implies the single crossing property (and vice versa). This result fits with the cbservation that someone who is risk neutral has a constant marginal utility, implying that a risk averse person(or a risk lover) has greater variation in his marginal utility.

It is natural, also, to consider the induced distributions of utility for different utility functions. We now have two degrees of freedom, one of which can be used to maintain expected utility when risk aversion increases. The other can be used to line up origins, say by $U(0,\rho_1) = U(0,\rho_2)$ or $U(1,\rho_1) = U(1,\rho_2)$. With either of these normalizations we have the single crossing property (as can be seen in figure 4 since $s = \phi(s)$ is unique, apart from zero, for $\phi$ concave)[12] Unfortunately for the seeker of intuitive content in this relationship, the two normalizations lead to opposite conclusions as to which utility distribution is riskier.

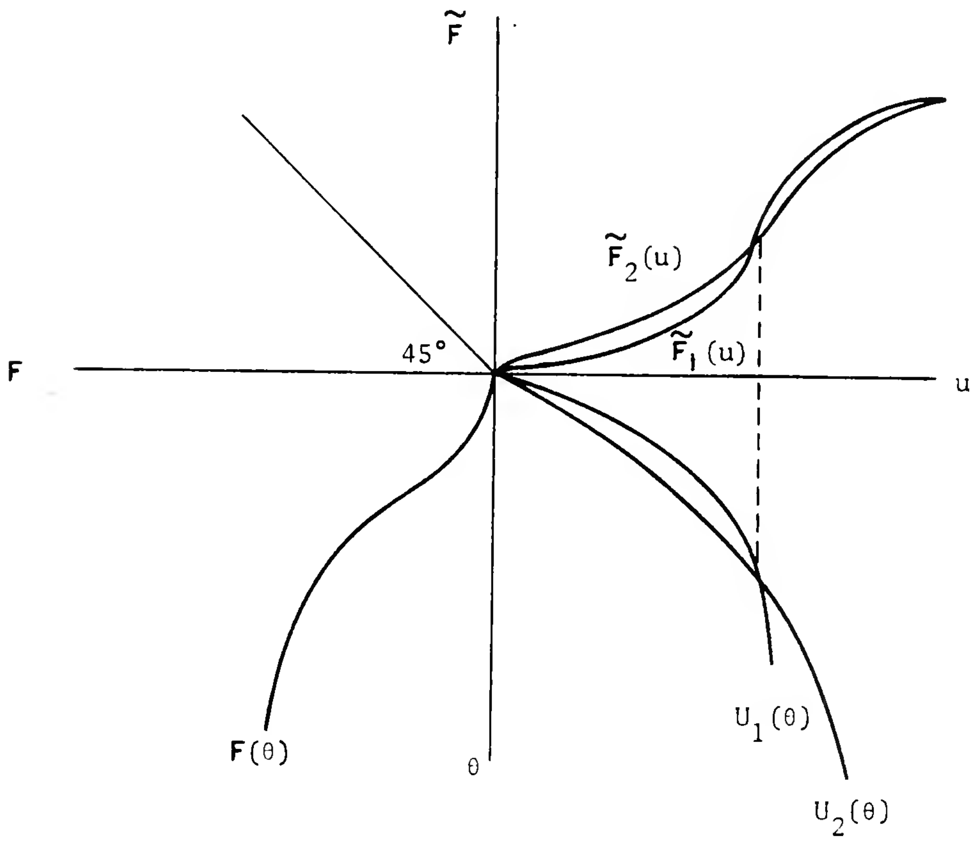


Figure 4

The formulation of the definitions of risk aversion are structured to cover the familiar single argument utility function, or the appearance of a single random variable in a many argument utility function. For example the two period consumption model with known wages, to be discussed below in section 8 , falls within this formulation, since the only random variable is the rate of return. In making comparisons between utility functions of several arguments, the logic of examining individuals who differ only in their degree of risk aversion is to assume that the two individuals being compared have the same indifference curves between random and control variables denoted by $\theta$ and $\alpha$ respectively. (These may be vectors rather than scalars as in this analysis.) Thus we have increased risk aversion if there is a monotone concave function $\phi$ so that

$$U_1(\theta,\alpha) = \phi(U_2(\theta,\alpha)) \tag{27}$$

Considering a family of utility functions indexed by $\rho$, the requirement of identical indifference curves implies that we can write the family of functions $V(\theta,\alpha,\rho)$ in the separable form $V(U(\theta,\alpha),\rho)$.The concavity condition gives us a derivative property (analogous to (19)).

$$\frac{\partial^2 \log V_U}{\partial U \partial \rho} < 0 \tag{28}$$

for risk aversion to increase with $\rho$.[13]

### 3.2. Consequences

The analysis above on the close relationship between increased risk and increased risk aversion suggests the possibility of an analogue to theorem 2 relating increases in the control variable to increased risk aversion rather than increased riskiness. In signing the effect of an increase in risk, we used a relationship between the control variable and risk aversion. To sign the effect of an increase in risk aversion we will use an assumption on the interaction between the control variable and the random variable. Let us state the result before examining the condition. (As above we assume U to be increasing in $\theta$.)

Theorem 4: Let $\alpha^*(\rho)$ be the level of the control variable which maximizes $\int V(U(\theta,\alpha),\rho)dF(\theta)$. If increases in $\rho$ represent increases in risk aversion (i.e., satisfy (23)), then $\alpha^*$ increases (decreases) with $\rho$ if there exists a $\theta^*$ such that $U_\alpha \geq (\leq)\ 0$ for $\theta \leq \theta^*$ and $U_\alpha \leq (\geq)\ 0$ for $\theta \geq \theta^*$.

Proof: The first order condition for optimality is

$$\int V_U U_\alpha dF = 0$$

By implicit differentiation we have

$$d\alpha^*/d\rho = -[\int V_{U\rho} U_\alpha dF]/[\int V_{UU} U_\alpha^2 + V_U U_{\alpha\alpha})dF] \qquad (29)$$

Concavity of V in $\alpha$ implies that the sign of $d\alpha^*/d\rho$ is the same as that of $\int V_{U\rho} U_\alpha dF$. Multiplying and dividing by $V_U$ and adding a constant times the first order condition, we have

$$\int V_{U\rho} U_\alpha dF = \int\left(\frac{V_{U\rho}(\theta)}{V_U(\theta)} - \frac{V_{U\rho}(\theta^*)}{V_U(\theta^*)}\right) V_U(\theta) U_\alpha(\theta) dF(\theta) \qquad (30)$$

The integrand is everywhere positive (negative) since both terms change sign just once at $\theta^*$. ||

The mathematical parallel between theorems 2 and 4 is most clearly brought out by considering the former for an increase in risk which has the single crossing property. Plotting the two terms to be multiplied in the integrand to sign the numerator we have

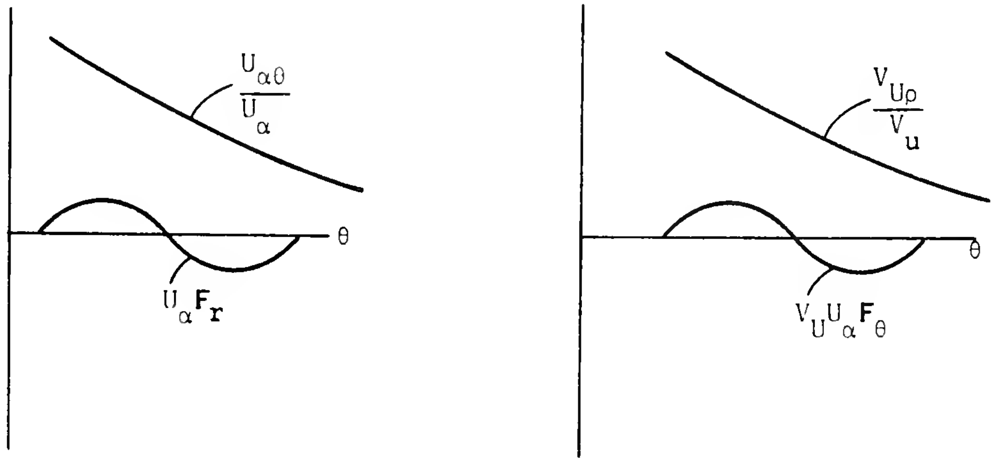


Figure 4

The result follows from the monotonicity of $\frac{U_{\alpha\theta}}{U_\alpha}\left(\frac{V_{U\rho}}{V_U}\right)$ and the fact that $U_\alpha F_r(V_U U_\alpha F_\theta)$ integrates to zero with a single sign change.[14]

Pursuing the parallel between the proofs, we could integrate the numerator in (29) by parts before seeking a sufficient condition to sign the expression:

$$\int V_{U\rho} U_\alpha dF = \int \frac{V_{U\rho}}{V_U} V_U U_\alpha dF = -\int_0^1 \frac{\partial^2 \ln V_U}{\partial\rho\partial\theta} \hat{T}(\theta) d\theta \tag{31}$$

where $\hat{T}(\theta) = \int_0^\theta V_U U_\alpha dF$ and using a normalization of V to preserve expected marginal utility in a similar fashion to the analysis in the previous section. We could now replace the single crossing property in theorem 4 by the nonnegativity of $\hat{T}$. However, if the single crossing property is not satisfied by the utility function the integral condition depends on the distribution of $\theta$ as well as the properties of V (and can be reversed in sign for some distribution of $\theta$). In addition we have not found applications of this more general assumption.[15]

The single crossing property of $U_\alpha$ as a function of $\theta$ may be satisfied from a variety of alternative assumptions on the structure of the choice problem. One way of ensuring a single crossing is to have $U_\alpha$ monotone in $\theta$ when it crosses zero, i.e., $U_{\alpha\theta}$ uniformly signed at $U_\alpha = 0$. Given continuity, two crossings would necessarily have opposite slopes. If we consider the certainty problem where $\theta$ is a parameter, then the sign of $U_{\alpha\theta}$ when $U_\alpha = 0$ is the same as the sign of the derivative of the optimal level of the control variable with respect to the parameter $\theta$.[16]

Corollary: Let $\hat{\alpha}(\theta)$ be the level of the control variable which maximizes $U(\alpha,\theta)$. Let $\alpha^*(\rho)$ be the level of the control variable which maximizes $\int V(U(\alpha,\theta),\rho)dF(\theta)$. If increases in $\rho$ represent increases in risk aversion (i.e., satisfy (23)), then $\alpha^*$ increases (decreases) with $\rho$ if $\hat{\alpha}$ decreases (increases) with $\theta$.

## 4. Choice of an income distribution

Let us think of individuals choosing a career as selecting a distribution of possible lifetime incomes. If we can order careers so that the single crossing property is satisfied for increases in the riskiness of a career then we have

Example I  More risk averse individuals choose less risky careers.

This follows from the third definition of increased risk aversion (Theorem 3) and the Corollary to Theorem 2. This result is interesting only as a confirmation of the matching of the definitions of risk and risk aversion, in the same way that we interpreted theorem 2 as saying that the correct response to an increase in risk is to adjust the control variable in the direction that would decrease risk aversion.

The first example used the parameter to look across individuals. For the second, let us consider the response of a given individual to an exogenous parameter change, by examining the impact of a proportional income tax on the level of career risk. Denoting the tax rate by t, career choice is the maximization over r of

$$\int_0^1 B((1-t)\theta)dF(\theta,r) \tag{32}$$

Identifying t with the shift parameter, $\alpha$, in the corollary to theorem 2 and identifying B as the utility function, we have

$$U(\alpha,\theta) = B((1-\alpha)\theta) \tag{33}$$

$$U_\theta = (1-\alpha)B', \quad \partial \log U_\theta/\partial\alpha = -((1-\alpha)^{-1} + \theta B''/B')$$

Thus the conditions of the corollary are satisfied if $xB''(x)/B'(x)$ is monotonic in x (which is equivalent to a uniform sign of $\partial^2 \log U_\theta/\partial\alpha\partial\theta$). The expression $-xB''/B'$ appears frequently in calculations describing behavior under uncertainty and has been named the measure of relative risk aversion. Since behavior in a variety of different circumstances can be related to measures of risk aversion, these serve as a method of organizing knowledge about the implications of behavior in certain circumstances for behavior in others. Let us note the result we have just shown before considering the widely used measures

Example 2 Increasing the rate of a proportional income tax increases (decreases) the riskiness of career choice if relative risk aversion is increasing (decreasing).

## 5. Measures of Risk Aversion

Three measures of risk aversion have received attention in the literature. We shall relate these to theorem 3 by examining the circumstances under which a variable can serve as an index of risk aversion as defined in that theorem. For different problems this will coincide with assumptions about these measures of risk aversion. Following Menezes and Hanson, we shall call them measures of absolute (A), relative (R) and partial (P) risk aversion[17] and define them for a utility function B(x) as

$$\begin{aligned} A(x) &= -B''(x)/B'(x) \\ R(x) &= -xB''(x)/B'(x) \\ P(x,y) &= -xB''(x+y)/B'(x+y) \end{aligned} \tag{34}$$

They are related by the equation

$$P(x,y) = R(x+y) - yA(x+y) \tag{35}$$

The key ingredient in the analysis, as[18] above, is whether the measures increase or decrease in x. Differentiating the definitions we have

$$\begin{aligned} A'(x) &= -(B'(x))^{-2}(B'(x)B'''(x)-(B''(x))^2) \\ R'(x) &= -B''(x)/B'(x)-x(B'(x))^{-2}(B'(x)B'''(x)-(B''(x))^2) \\ P_x(x,y) &= -B''(x+y)/B'(x+y)-x(B'(x+y))^{-2}(B'(x+y)B'''(x+y)-(B''(x+y))^2) \end{aligned} \tag{36}$$

with the obvious relationship

$$P_x(x,y) = R'(x+y) - yA'(x+y) \tag{37}$$

Since the assumption of an increasing, concave utility function has no sign implications on the third derivative of the utility functions over a finite range, it is clear that absolute risk aversion may increase or decrease. The same is true for the other indices only if x does not assume the value zero. This is not a restriction for relative risk aversion, since most problems are set up to exclude zero consumption. It does represent a restriction on partial risk aversion since the first order condition for many problems will require x to take on both positive and negative values.

To relate the three measures of risk aversion to the analysis of section 3, we must consider some specific problem (as we did in example 2) and select some variable of the problem to serve as an index of risk aversion. For differently structured problems the sign conditions on the index of risk aversion ( $\partial^2 \log U/\partial\theta\partial\rho$) will be equivalent to an increasing or decreasing measure of risk aversion for one of the three measures. To relate the three expressions in (36) to our index we can ask what interaction between $\theta$ and $\rho$ will yield a one signed derivative in (36) for a one signed index of risk aversion. By straight forward calculation one can check the three formulations

$$\begin{aligned} U(\theta,\rho) &= B(\theta+\rho) \\ U(\theta,\rho) &= B(\theta\rho) \\ U(\theta,\rho) &= B(y+\theta\rho) \end{aligned} \tag{38}$$

Thus we can conclude that absolute risk aversion increases when leftward translation of the axis corresponds to a concave transform of the utility function. Similarly relative risk aversion increases when multiplicative stretching of the axis(about the origin) corresponds to a concave transform. Increasing partial risk aversion corresponds to a concave transform for multiplicative transform of the axis beyond the value y.

To illustrate the role of the measures of risk aversion, let us examine theorem 3 which showed that an increasing index of risk aversion was equivalent to an increased risk premium. The definition of the risk premium was

$$\int U(\theta,\rho)dF = U(\int\theta dF - p(\rho),\rho). \tag{39}$$

We shall examine the relationship of the risk premium to initial wealth **w** and the size of the gamble, $\tilde{z}$. For this purpose we define the risk premium alternatively as

$$\int U(w + z)dF(z) = U(w+\int zdF(z) - \pi(w,z)) \tag{40}$$

$$\text{or} \quad -\pi(w,z) = U^{-1}(\int U(w+z)dF(z)) - \int zdF - w$$

To take advantage of the special forms of the family of utility functions, we can allow the index to affect just wealth, or wealth and the size of the gamble, or just the size of the gamble.

For these formulations (equation 38) respectively we make the three substitutions

$$\begin{aligned} &z \text{ for } \theta \quad \rho w \text{ for } \rho \\ &w + z \text{ for } \theta \\ &z \text{ for } \theta \quad w \text{ for } y. \end{aligned} \tag{41}$$

Substituting in the definition of the risk premium (39) for the three formulations we have.

$$\begin{aligned} &\int B(z + \rho w)dF(z) = B(\int zdF(z) - p(\rho) + w\rho) \\ &\int B((w + z)\rho)\, dF(z) = B((w+\int zdF(z) - p(\rho))\rho) \\ &\int B(w+\rho z)dF(z) = B(w+(\int zdF(z) - p(\rho))\rho) \end{aligned} \tag{42}$$

Solving these three equations for $p(\rho)$ we have

$$\begin{aligned} -p(\rho) &= B^{-1}(\int B(z + \rho w)dF) - \int zdF - w\rho \\ -\rho p(\rho) &= B^{-1}(\int B((w+z)\rho)dF) - \int \rho zdF - w\rho \\ -\rho p(\rho) &= B^{-1}(\int B(w+\rho z)dF) - \int \rho zdF - w \end{aligned} \tag{43}$$

Comparing (43) and (40) we can relate the risk premium as a function of wealth and gamble size to its definition in terms of the risk index

$$p(\rho) = \begin{cases} \pi(\rho w, z) \\ \pi(\rho w, \rho z)/\rho \\ \pi(w, \rho z)/\rho \end{cases} \tag{44}$$

Thus, by theorem 3 we have shown that

$$\begin{aligned} &\partial\pi(\rho w, z)/\partial\rho \gtrless 0 &&\text{as} \quad A'(x) \gtrless 0 \\ &\partial(\pi(\rho w, \rho z)/\rho)/\partial\rho \gtrless 0 &&\text{as} \quad R'(x) \gtrless 0 \\ &\partial(\pi(w, \rho z)/\rho)/\partial\rho \gtrless 0 &&\text{as} \quad P_x(x,w) \gtrless 0 \end{aligned} \tag{45}$$

## 6. Portfolio

The problem which has received the most attention in this general area is the division of a given initial wealth between safe and risky assets. Denoting initial wealth, security holdings, and the rate of return on the safe and risky assets by w, s, m and i, we can write utility as

$$B(w(1+m) + s(i-m))$$

The most obvious application of the results above comes from considering a family of utility functions showing increased risk aversion. Then, by theorem 4[19]

Example 3:[20] Of two individuals with the same wealth, the more risk averse has a lower absolute value of security holding.

The focus of much of the attention in this area has been on the implications of changing initial wealth for security holdings. As in the previous section we can obtain the wellknown relationships of the derivatives of security holding to wealth as corollaries to example 3 by identifying w with the index of risk aversion.

To match the notation of this section with that of theorem 4, let us pair $(w(1+m), s(i-m))$ with $(\rho,\alpha,\theta)$ and set $U(\alpha,\theta)$ equal to sr, the (random) return on security holdings; thus writing $V(U,\rho)$ as $B(w(1+m) + U)$. Then $-V_{UU}/V_U$ equals the index of absolute risk aversion. Thus we have

**Corollary 1:** The absolute value of security holdings increases(decreases) with wealth if absolute risk aversion decreases (increases) with wealth.

One aspect of the approach we have taken is that the index of relative risk aversion also arises from the concavity property upon examining the fraction of wealth held in risky securities, which we denote by $\hat{s}$. We now write utility as

$$B(w(1+m + \hat{s}\,(i-m)))$$

If we identify $(w,\hat{s},i-m)$ with $(\rho,\alpha,\theta)$ and set $U(\alpha,\theta)$ equal to $1+m+\hat{s}(i-m)$ the gross rate of return on total wealth, then $V(U,\rho)$ becomes $B(w\ U)$. The concavity condition, $-\partial^2 \ln V_U/\partial U\partial\rho$, is equal to the derivative of the index of relative risk aversion. Thus we have

Corollary 2. The absolute value of the fraction of security holdings increases (decreases) with wealth if relative risk aversion decreases (increases) with wealth.

We have now considered three applications of theorem 4 relating choice to risk aversion. Now let us turn to the effects of change in the distribution of the random variable.

Example 4:[21] A mean utility preserving increase in risk decreases the absolute value of security holdings if $P_x(\alpha\theta,w) > 0$.

Identifying $(s, i - m)$ with $(\alpha,\theta)$, it is straight forward to check that $P_x$ and condition (15) of theorem 2 have opposite signs. From the relations among risk aversion indices, one can see that a sufficient condition for $P_x$ to be positive is that absolute risk aversion be decreasing and relative risk aversion increasing. Thus we have

Corollary If the consumer has decreasing absolute risk aversion and increasing relative risk aversion, then the absolute value of security holdings decreases with a mean utility preserving increase in risk.

## 7. Taxation and Portfolio Choice

One form of change in return distribution which is easily parametrized is that induced by tax change.Tax structures vary in their loss offset provisions and in the tax rates on the returns to different assets (e.g. capital gains as opposed to interest income). There are two causes of demand changes which raise obvious questions - changes in tax rates and in offset provisions. We assume that holdings of each asset and the safe rate of return are non-negative ( $m \geq 0$, $s \geq 0$, $w-s \geq 0$ ) We begin with the cases of full and no loss offsets, before turning to more complicated partial offsets. To explore tax rate changes, we shall write utility as a function of the tax rate with a given distribution of before tax returns and examine demand changes by exploring taxes as indices of risk aversion. To compare tax structures we shall consider changes in the after tax rate of return.

We denote the tax rates on the two incomes by $t_i$ and $t_m$. With full loss offset we can write utility of after tax terminal wealth as

$$B(w + si(1-t_i) + (w-s)m(1-t_m)) \tag{46}$$

With no loss offset, we would write utility as

$$B(w + s\text{Min}(i(1-t_i),i) +(w-s)m(1-t_m)) \tag{47}$$

It is straight forward to check that the partial risk aversion measure indicates whether $t_i$ is an index of risk aversion. Thus we have

Example 5:[22] Risky security holdings increase with the tax rate on the return to the risky asset with full loss offset if partial risk aversion is increasing, for all i in the relevant range, i.e. if $P_x(x,y) > 0$ with $x = si(1-t_i)$ and $y = w + (w-s)\ m(1-t_m)$. The same proposition holds with no loss offset under the slightly weaker condition that $P_x$ is positive throughout the relevant range of positive rates of return, i.

To compare the different offset rules, let us denote the after tax return by j and write utility as

$$B(w+sj+(w-s)m(1-t_m))$$

If the distribution of the before tax return is F(i), with full loss offset the distribution of after tax return satisfies

$$G(j,t^f) = F(j/(1-t^f)). \tag{48}$$

where we distinguish tax rates in the two cases by $t^f$ and $t^n$. With no loss offset the distribution of after tax return satisfies

$$H(j,t^n) = \begin{array}{ll} F(j) & j \leq 0 \\ F(j/(1-t^n)) & j \geq 0 \end{array} \tag{49}$$

If the two tax structures are to yield the same expected utility $t_f$ must exceed $t_n$. Thus this change in tax structure and rates satisfies the single crossing property, **with** the loss offset giving the less risky distribution

Example 6: If partial risk aversion is increasing a tax on the risky asset with full loss offset results in a higher level of security holdings than a tax with no loss offset and a lower tax rate to result in equal expected utility. In addition, expected tax revenue is higher with a full loss offset in this case.

To prove the second part of the example, it is sufficient to show that expected government revenue per unit invested is higher with full loss offset and these tax rates.

If the tax rates were adjusted to keep expected revenue per unit of investment (and thus the expectation of j) constant, the change would be mean preserving rather than mean utility preserving. Thus expected utility would be lower with the riskier distribution, the case with no loss offset. To equate expected utilities, the tax rate without loss offset must be lowered, reducing expected revenue per unit invested.

To consider limited loss offsets, let us assume that both income sources are taxed at the same rate and losses on the risky asset may be offset against returns on the safe asset. In addition we shall allow partial offsetting of any remaining losses against initial wealth (or equivalently safe non-investment income). Let us denote by Y the level of before tax income

$$Y = si + (w-s)\,m. \tag{50}$$

The distribution of Y depends on the distribution of i and the choice of s

$$G(Y;s) = F((Y-(w-s)m)/s) \tag{51}$$

Let us note that

$$G_s = F'((wm-Y)/s^2) \tag{52}$$

so that the distribution satisfies the single crossing property for the Corollary to Theorem 2. Let us assume that some fraction of remaining losses, f, $0 \leq f \leq 1$, can be set off against initial wealth. Then, we can write utility as

$$B(w_o + \mathrm{Min}(Y(1-t),\ Y(1-ft)) \tag{53}$$

We can now examine the response to changes in f and t. By the Corollary to Theorem 2 we have

Example 7: With partial loss offset, risky security holdings increase with the tax rate and the offset fraction if partial risk aversion is increasing.

Since the tax rate and offset fraction must both increase to keep expected utility constant, it is clear that risky security holdings increase in this case too, provided that $P_x$ is positive.

## 8. Savings with a Single Asset

Consider an individual dividing his initial wealth, w, between current consumption, c, and savings receiving a rate of return, i. In the certainty problem he chooses c to maximize his two period utility function B(c, (w - c)(1 + i)). If his savings are an increasing function of the interest rate we have

$$\frac{\partial c}{\partial i} = -B_{ci}/B_{cc} < 0 \qquad (54)$$

Thus $B_{ci}$ is also negative. Now let us consider the return to be random. Then, in terms of our previous notation c is $\alpha$ and i is $\theta$. Thus

$$\begin{aligned} U_\alpha &= B_c \\ U_{\alpha\theta} &= B_{ci} > 0 \end{aligned} \qquad (55)$$

Thus the conditions of theorem 4 are satisfied and we have

Example 8: Savings decrease (increase) with increases in risk aversion if savings under certainty are an increasing (decreasing) function of the interest rate.

This result can be connected with the notion of a risk premium. Presumably more risk averse individuals have larger risk premiums. If we describe savings decisions under uncertainty as if they were made under certainty with risk premiums deducted from the expected return, then we would expect increased risk aversion to have the same effect as a decreased rate of return in the certainty problem.

To examine the response of savings to increased risk let us consider the special case where attitudes toward risk in each period are independent of the level of consumption in the other period. This situation has been described by R. Keeney and R. Pollack and requires a utility function of the form

$$B(x_1,x_2) = C_0 + b_1C_1(x_1) + b_2C_2(x_2) + b_3C_1(x_1)C_2(x_2) \tag{56}$$

In this case, the index of relative risk aversion for gambles in the second period depends only on $x_2$:

$$R_2(x_2) = -x_2B_{22}/B_2 = -x_2C_2''/C_2' \tag{57}$$

We can now state

Example 9: A mean utility preserving increase in risk increases (decreases) savings as relative risk aversion in period two is decreasing (increasing) with consumption.

Calculating the derivative of $R_2$ we have

$$R_2' = -C_2''/C_2' - x_2(C_2'C_2''' - C_2''C_2'')/C_2'C_2' \tag{58}$$

Identifying $\alpha$ with savings and $\theta$ with one plus the interest rate, we have

$$U(\alpha,\theta) = C_0 + b_1C_1(w - \alpha) + b_2C_2(\alpha\theta) + b_3C_1(w - \alpha)C_2(\alpha\theta) \tag{59}$$

Thus differentiating (59) we can express the condition of theorem 2 as:

$$U_\theta U_{\alpha\theta\theta} - U_{\alpha\theta}U_{\theta\theta} = \alpha^2(b_2 + b_3C_1)^2[C_2'C_2'' + \alpha\theta(C_2'C_2''' - C_2''C_2'')] \tag{60}$$

Clearly (58) and (60) have opposite signs.

## 9. Costs of Meeting Random Demand

Consider a firm with a concave production function $F(K,L)$ (with positive marginal products) which selects $K$ ex ante and $L$ ex post to meet a random demand $Q$ in order to maximize the expectation of utility of costs $\int B(rK + wL)dF(Q)$. We can prove the proposition[23]

Example 10: If capital and labor are complements ($F_{KL} > 0$). the more risk averse the firm the greater its level of capital.

Define $L(Q,K)$ implicitly by $Q = F(K,L)$. Then, identifying $(\alpha,\theta)$ with $(K,Q)$ we can check the condition of theorem 4 that $U_\alpha = B'(r + w \frac{\partial L}{\partial K})$ has a unique value of $Q$ for which it equals zero. This follows from the monotonicity of $\frac{\partial L}{\partial K}$ in $Q$, given $F_{KL} > 0$.

$$\frac{\partial L}{\partial K} = -\frac{F_K}{F_L}, \quad \frac{\partial^2 L}{\partial K \partial Q} = F_{LL}F_K F_L^{-3} - F_{LK}F_L^{-2} > 0 \tag{61}$$

To examine the effect of an increase in risk let us consider the special case of an expected cost minimizer with a constant returns to scale, constant elasticity-of-substitution production function.

Example 11: A mean utility (i.e., cost) preserving increase in risk increases (decreases) the level of capital if the elasticity of substitution is greater (less) than one.

Let $Q/K = F(1, L/K) = f(\ell)$. The elasticity of substitution $\sigma$, is given by

$$\sigma = \frac{-f'(f - \ell f')}{ff''\ell} \quad .$$

Let $a = \frac{f - f'\ell}{f'\ell}$, then, making the obvious identifications

$$U_\theta = w\frac{dL}{dO} = wF_L^{-1} = wf'^{-1}$$

$$U_{\alpha\theta} = wf''fK^{-1}f'^{-3} = aw\sigma^{-1}K^{-1}f'^{-1}$$

$$U_{\theta\theta} = -wf''K^{-1}f'^{-3}$$

$$U_{\alpha\theta\theta} = \frac{aw}{\sigma}(f''f'^{-3}K^{-2}) - \frac{w}{Kf'}\frac{d(a/\sigma)}{dQ}$$

Thus we have

$$U_\theta U_{\alpha\theta\theta} - U_{\alpha\theta}U_{\theta\theta} = \frac{w^2}{Kf'^2}\frac{d(a/\sigma)}{dQ} = \frac{w^2}{Kf'^2}\frac{d(a)}{d\ell}\left(\frac{1}{Kf'}\right)$$

With $\sigma$ constant a decreases (increases) with $\ell$ when $\sigma$ is greater (less) than one.

## 10. Hedging

It may be useful to examine a problem for which it is not generally valid that utility is monotonic in the random variable. Consider an individual who can purchase commodities today at fixed prices. Before he will consume, the market will reopen and permit trading at a price which, from the present standpoint is random. Let us denote by $x_1$, $x_2$, the current purchase and by $y_1$, $y_2$ the final consumption of both commodities. Let p be the current (determinate) price and q the future random price of good $2^{24}$. Then we can write the consumer's maximization problem as

$$\max_{x,y} \int B(y_1(q),y_2(q))dF(q) \tag{62}$$

$$\text{subject to: } y_1(q) + qy_2(q) = x_1 + qx_2 = w \tag{63}$$

$$x_1 + px_2 = 1 \tag{64}$$

$$y_1 \geq 0, \ y_2 \geq 0$$

We can analyze the choice of x by introducing the indirect utility function V, defined by

$$V(p,w) = \max_{y} B(y_1,y_2) \text{ subject to} \tag{65}$$

$$y_1 + py_2 = w$$

It is well known that

$$V_p = -y_2V_w \tag{66}$$

Substituting (63) and restating the basic problem we have

$$\max_{x} \int V(q, x_1 + qx_2)dF(q)$$

$$\text{subject to} \quad x_1 + px_2 = 1 \tag{67}$$

or

$$\max_{x_2} \int V(q, 1 - px_2 + qx_2)dF(q)$$

If we now examine the response of utility to the random price we have

$$\frac{dV}{dq} = V_p + x_2V_w = (x_2 - y_2)V_w \tag{68}$$

Not surprisingly, increases in price represent increases in utility only if the consumer will be a net supplier of good 2 at the market price. In general, as the diagram illustrates, the sign of $x_2 - y_2$ will depend on q.

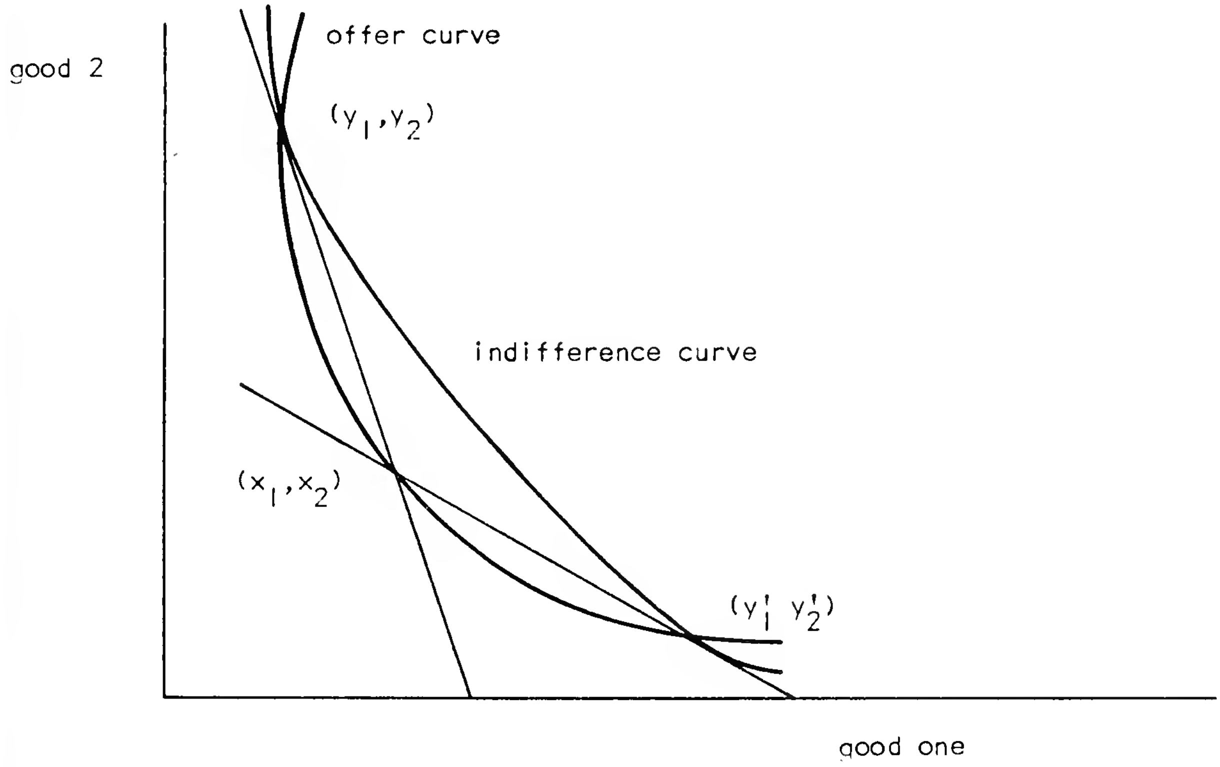

good 2
offer curve
$(y_1, y_2)$
indifference curve
$(x_1, x_2)$
$(y_1', y_2')$
good one

Figure 5

Considerations of consumer choice indicate that V will be monotonic in q for a given x in two (basically equivalent) circumstances - if the initial position is not in the consumption possibility set ($x_1 \leq 0$ or $x_2 \leq 0$) or if the indifference curves become horizontal at a level of consumption less than the endowment position. One obvious example of the latter is right angled indifference curves arising from the utility function

$$B(y_1,y_2) = C[\mathrm{Min}(y_1,y_2\delta^{-1})],V(p,w) = C(\frac{w}{1+p\delta}) \qquad (69)$$

We shall say a consumer has a purely speculative position if he holds a negative quantity of one commodity (i.e. has gone short). With the utility function(69) we can define the pure hedging position $\hat{x}$ as the quantity vector arising from the maximization of certainty utility at the ex ante prices.[25]

Example 12 : If two individuals with the same wealth have purely speculative positions, i.e. both have short positions in one commodity, then the less risk averse has speculated more, i.e. holds less algebraically of the good with the short position.

Example 13 : Given the utility function of (69) of two individuals with the same wealth, the less risk averse speculates more in the sense that $|x_1 - \hat{x}_1|$ is larger.

In both examples we have sufficient assumptions to ensure the monotonicity of U in θ. It is straightforward to check that the single crossing property of theorem 4 is satisfied.

Considering a change in risk we have the proposition

Example 14: For a consumer with the utility function (69), a mean utility preserving increase in risk increases (decreases) speculation, i.e., $|x_1 - \hat{x}_1|$, as $P_x(y_1 - \hat{x}_1, \hat{x}_1) < (>) 0$.

Identifying $(\alpha,\theta)$ with $(x_2,q)$ we can write

$$U(\alpha,\theta) = V(\theta, 1 + (\theta - p)\alpha) = C((1 + (\theta - p)\alpha)/(1 + \theta\delta)) \tag{70}$$

Differentiating often enough, we have

$$\begin{aligned}&(U_\theta U_{\alpha\theta\theta} - U_{\alpha\theta}U_{\theta\theta})(1 + \theta\delta)^6(\alpha - (1 - p\alpha)\delta)^2(1 + p\delta)^{-1} \\ &\quad = C'C'' + (1 + \theta\delta)^{-1}(1 + p\delta)^{-1}(\alpha - (1 - p\alpha)\delta)(\theta - p)(C'C''' - C''C''). \\ &\quad = -(C')^2 P_x(y_1 - \hat{x}_1, \hat{x}_1)\end{aligned} \tag{71}$$

## Footnotes

No.

[*] An earlier version of this paper was presented at the NSF-NBER Conference on Decision Rules and Uncertainty at the University of Iowa, May 1972. The authors are indebted for many helpful comments received there. Financial support by the National Science Foundation and the Ford Foundation are gratefully acknowledged.

[1] This statement included the possibility that the two individuals being compared are the same individual with different level of some parameter, such as wealth.

[2] $\int_0^1 \theta dG - \int_0^1 \theta dF = \int_0^1 \theta dS = -\int_0^1 S d\theta = \theta S(\theta)\Big]_0^1 = 0$, applying integration by parts, equation (2) and the obvious property $S(0) = S(1) = 0$.

[3] $F$ is assumed to be twice continuously differentiable with respect to $\theta$ and $r$.

[4] The yield-compensated change in yield of a security, considered by P. Diamond and M. Yaari, is an example of a mean utility preserving increase in risk. They observed that such a change had no "substitution effect" - only "income effects", in the sense that the impact on security holdings was similar to the impact of a change in the distribution of non-investment income.

[5] For many problems the positivity of $U_\theta$ for all $\alpha$ will follow naturally, as with investment with a random return. However with the possibility of short sales $U_\theta$ may be positive for some $\alpha$ and negative for others depending on the position taken by the investor. Also $U_\theta$ may not be of one sign for a given $\alpha$.

No.

6 For arbitrary changes in the distribution of $\theta$ one can consider dividing the total change in a fashion analogous to the Slutsky equation. Thus one would subtract a change in the distribution which had the same impact on expected utility, leaving a mean utility preserving change, which might have a signed impact on $\alpha$ if it represents an increase in risk. The subtracted change could be divided into income and substitution effects. This approach was taken by Diamond.

7 The theorem is also valid for a discrete change in riskiness such that expected utilities at the respective optima are equal. Since expected marginal utility is decreasing in the control variable ($U_{\alpha\alpha} < 0$ is assumed) the proof follows from determining the sign of $\int U_{\alpha}(\theta,\alpha(r_1))dF(\theta,r_2)$ in a parallel fashion to the proof given.

8 An alternative proof can be constructed by substituting $U^{-1}(u,\alpha)$ for $\theta$ in the first order condition and applying the proof of theorem 1.

9 This interpretation was suggested by James Mirrlees.

10 For the present, we suppress the role of the control variable $\alpha$. In some problems it will not appear. In others we can apply Theorem 3 for a given level of $\alpha$.

11 We ignore complications in the statement of these conditions arising on sets of values of $\theta$ at measure zero.

12 There must be one solution (apart from zero) since expected untilities are the same.

No.

13 Given our assumption that $U_\theta$ is positive, this does not represent a change in conditions since $\partial^2 \log V_\theta / \partial\theta\partial\rho = U_\theta \partial^2 \log V_U / \partial U \partial\rho$. These conditions remain equivalent to $\int V_\rho F_{\theta r} < 0$ and $p^{\prime}(\rho) > 0$ where $V(U(\int\theta dF - p,\alpha)\rho) =$

$$\int V(U(\theta,\alpha),\rho)dF$$

14 This proof points up the fact that the Corollary to Theorem 2 is also derivable as a corollary to Theorem 4, letting utility, U, be the random variable, which implies that the control variable, $\alpha$, affects the distribution of the random variable.

15 This is the same situation as with the Corollary to Theorem 2.

16 We are indebted to R. Khilstrom and L. Mirman for pointing out a deficiency in our earlier proof.

18 The measures indicate aversion to small risks at a given point. To evaluate responses to a large risk, we need assumptions on the measures throughout the relevant range of income.

17 A and R were introduced by Arrow and J. Pratt. P was introduced by C. Menezes and D. Hanson whose approach and notation we follow and whose results we relate to our approach, and R. Zeckhauser and E. Keeler.

19 Clearly $\partial B/\partial s = (i-m)B' = U_\alpha$ changes sign only once. $U_\theta \; {}^{\partial B}/\partial i = sB' = U_\theta$ is signed by the sign of s. Since a negative value of $U_\theta$ reverses the sign of the effect of $\rho$ on $\alpha^*$ in theorem 3, we find that s decreases or increases as it is positive or negative. Thus the absolute level of security holdings, $|s|$ decreases.

No.

20 This result can also be viewed as a corollary to Example 1. Pairing $(w(1+m), s, (i-m)s)$ with $(\alpha, r, \theta)$ and writing utility as $B(\alpha + \theta)$ we have the distribution of return $F(\theta, r)$ equal to $G(\theta/r)$. Since $F_r$ equals $-\theta r^{-2} G'$, we have the single crossing property and can apply the corollary to theorem 2.

21 Since $(i-m)$ must change sign to satisfy the first order condition, $P_x$ negative through the relevant values of $\theta$ is not possible.

22 This example shows clearly the limitation on the possibility of a negative $P_x$. With $m = 0$ and full loss offset it is clear from the utility function that $s^*(1-t_i)$ is constant independent of the shape of B. Thus the reversed conclusion of the example would never occur for this case.

23 If there are constant returns to scale, $F_{KL}$ is always positive.

24 This model has been examined by Stiglitz (1970)

25 $\hat{x}_1$ is equal to $1/(1 + p\delta)$, $\hat{x}_2$ is equal to $\hat{x}_1 \delta$.

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